Eigencentrality in bipartite graphs (QSIURP 2018-2019)

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Centrality

A measure for how central each node in a graph is.

- **Degree**: number of neighbors
- **Closeness**: least average shortest path length
- **Betweenness**: occurs on how many shortest path
- **Eigen-centrality**: value in normalized principal eigenvector

Eigen-centrality

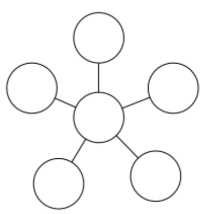
- No direct geometric interpretation.
- A node's importance is a function of the importance of its neighbors.
- Used by Google to rank webpages.

Centralization

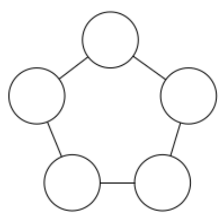
A measure for how centralized a graph is.
We use the following definition:

$$C(G) = \sum_{v \in V(G)} c^* - c_v$$

i.e. the sum of differences between centralities of every vertex and c^*, the most central node



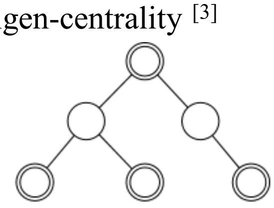
5-star
strong centralization



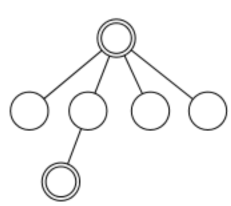
5-cycle
weak centralization

Related Work

- The star graph is known to have the maximal centralization score for all measures above.
- For bipartite graphs with fixed size parts, the generalized star has maximal centralization for degree, betweenness [1] and closeness [2].
- It is conjectured to have highest centralization for eigen-centrality [3]



(4,2)-generalized star
(2,1,1,1)-star



(2,4)-generalized star
(4,1,3,0)-star

Empirical Results

- Generated all small bipartite graphs using `nauty`
- Computed eigen-centralization with `networkx`
- Result agree with conjecture

Analytical Results

Definition

(n,d,c,i)-star: a graph with n nodes attached to a center, c of them connected to i, the rest connected to $d+i$.

Theorem

The $(n,d,c,0)$ has principal eigenvalue λ^2 and eigenvector $[\mathbf{c}_A \ \mathbf{c}_B]$ corresponding to the two parts of the graph given by

$$\lambda^2 = \frac{n + d + \sqrt{(n+d)^2 - 4cd}}{2}$$

$$\mathbf{c}_B = \Bigg[\overbrace{\frac{n + d - 2c \pm \sqrt{(n+d)^2 - 4cd}}{2(n-c)}, \ldots}^{n-c}, \underbrace{1, \ldots}_{c} \Bigg]$$

$$\mathbf{c}_A = \frac{1}{\lambda} M \, \mathbf{c}_B = \frac{1}{\lambda} \left[\sum \mathbf{c}_B, \ \underbrace{X \ldots}_{d(n-c)} \ \underbrace{1 \ldots}_{c(d-1)} \right]^T$$

Conjecture

The some of the squares of the entries corresponding to each part of a bipartite graph in the principal eigenvector are equal

Remark

For the graph of a tree, the following bound holds

$$2 - \frac{1}{m} \le \lambda \le \sqrt{m + n - 1}$$

References

[1] Philip Sinclair. “Betweenness Centralization for Bipartite Graphs”. In: The Journal of Mathematical Sociology 29.1 (2004), pp. 25–31.

[2] Matjaž Krnc, Jean-Sébastien Sereni, Riste Škrekovski, and Zelealem B. Yilma. “Closeness centralization measure for two-mode data of prescribed sizes”. In: Network Science 4.4 (2016), pp. 474–490.

[3] Martin G. Everett, Philip Sinclair & Peter Dankelmann (2004) SOME CENTRALITY RESULTS NEW AND OLD, The Journal of Mathematical Sociology, 28:4, 215-227.

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