Sequoia: a Playground for Logicians

Meta-properties

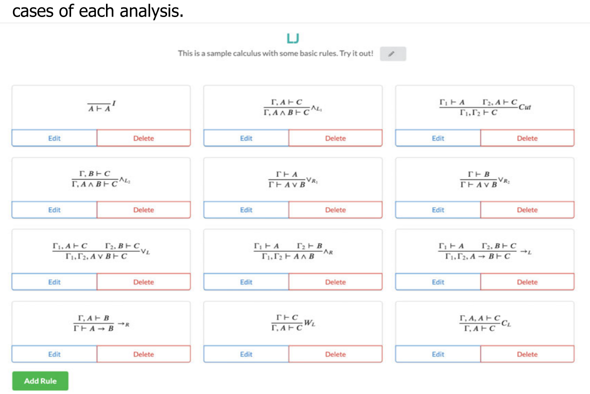
Problem

Sequent calculus is a pervasive technique for studying logics and their properties due to the regularity of rules, proofs, and meta-property proofs across logics. However, composing these proofs brings up a variety of issues:

- Simple proofs can be large and writing them by hand is often messy
- Formatting the proofs to Latex can be tedious and confusing
- The proofs of meta-properties involve building and verifying several proof trees to cover all the cases, which can be tedious.

Solution

Sequoia is a web-based application for specifying sequent calculi and performing basic reasoning about them. We provide an intuitive interface where inference rules can be input in LaTeX and are immediately rendered with the corresponding symbols. We also provide checks for some of the most important meta-theoretical properties, such as weakening admissibility and identity expansion, given that they proceed by the usual structural induction. In this sense, the logician is only left with the tricky and most interesting cases of each analysis.



Sequoia currently performs checks for identity expansion, weakening admissibility, permutability, and cut-elimination. Each property requires checking several general cases to find a proof. Examples for such cases will use one or more of the following rules:

$$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge_R \qquad \dfrac{\Gamma \vdash A \quad \Gamma, A \vdash C}{\Gamma \vdash C}\mathit{cut}$$

$$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_L \qquad \dfrac{}{\Gamma, A \vdash A}\mathit{id}$$

Identity Expansion

Identity expansion is the property that all identity rules in a proof tree are applied on atoms and not complex formulas such as $A \wedge B$. The proof proceeds by induction on the structure of the formula the identity rule is applied on. This is done by checking that when the identity rule is applied on a complex formula such as $A \wedge B$, the identity rule can be replaced by identity rules on the subformulas A and B. Each connective defined in the calculus is a case that needs to be checked.

$$\dfrac{}{\Gamma, A \wedge B \vdash A \wedge B}\mathit{id} \rightsquigarrow \dfrac{\dfrac{\dfrac{}{\Gamma, B, A \vdash A}\mathit{id} \quad \dfrac{}{\Gamma, A, B \vdash B}\mathit{id}}{\Gamma, A, B \vdash A \wedge B}\wedge_R}{\Gamma, A \wedge B \vdash A \wedge B}\wedge_L$$

Weakening Admissibility

The Weakening admissibility property states that for a context Γ, if the sequent $\Gamma \vdash \Delta$ has a proof, then the sequent $\Gamma, F \vdash \Delta$ also has a proof. This property can be checked separately for any possible context and the usual proof proceeds by structural induction on the derivation of S. Currently, Sequoia can check all "trivial" cases, i.e. the ones that require only the inductive hypothesis.

Permutability

Given two rules R_1 and R_2 (such as $\wedge_R$ and $\wedge_L$), the initial rules, and the weakening properties of the calculus, we say that R_1 permutes up R_2 if a proof tree T ending with the rule R_2 applied over R_1 can be transformed into a proof tree T' ending with R_1 applied over R_2. Sequoia performs this check as follows:

1) Generating all derivations T where R_2 ($\wedge_R$) is applied over R_1 ($\wedge_L$)

$$\dfrac{\dfrac{\Gamma, A, B \vdash C \quad \Gamma, A, B \vdash D}{\Gamma, A, B \vdash C \wedge D}\wedge_R}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_L$$

2)Generating all derivations T' where R_1 ($\wedge_R$) is applied over R_2 ($\wedge_L$)

$$\dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_L \quad \Gamma, A \wedge B \vdash D}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_R \qquad \dfrac{\Gamma, A \wedge B \vdash C \quad \dfrac{\Gamma, A, B \vdash D}{\Gamma, A \wedge B \vdash D}\wedge_L}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_R$$

$$\dfrac{\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_L \quad \dfrac{\Gamma, A, B \vdash D}{\Gamma, A \wedge B \vdash D}\wedge_L}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_R$$

3)Then, for each tree T generated in step 1, we try to find a tree T' generated in step 2 such that T can be transformed to T'. In this case, we have

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, A, B \vdash C} \quad \overset{\mathcal{E}}{\Gamma, A, B \vdash D}}{\Gamma, A, B \vdash C \wedge D}\wedge_R}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_L \rightsquigarrow \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma, A, B \vdash C}}{\Gamma, A \wedge B \vdash C}\wedge_L \quad \dfrac{\overset{\mathcal{E}}{\Gamma, A, B \vdash D}}{\Gamma, A \wedge B \vdash D}\wedge_L}{\Gamma, A \wedge B \vdash C \wedge D}\wedge_R$$

Cut-Elimination

The proof we use for cut elimination uses Gentzen's proof strategy, where we try to use structural induction to prove that a proof tree using cut can be transformed into a proof tree without any cut rules. The proof uses structural induction on the ordered triple (C,l,r), where C is the cut formula, l is the left subtree, and r is the right subtree. The base cases are cases where a cut rule is applied, and an axiom rule is applied to one of the premises of the resulting tree (i.e. l or r is empty). For each of these cases, we try to close the proof without using cut by either applying the axiom to the sequent cut is applied to[1] or using the proof of a premise instead of applying cut[2].

(1) $$\dfrac{\overset{\mathcal{D}}{\Gamma, A \vdash B} \quad \dfrac{}{\Gamma, B, A \vdash A}\mathit{id}}{\Gamma, A \vdash A}\mathit{cut} \rightsquigarrow \dfrac{}{\Gamma, A \vdash A}\mathit{id}$$

(2) $$\dfrac{\overset{\mathcal{D}}{\Gamma \vdash A} \quad \dfrac{}{\Gamma, A \vdash A}\mathit{id}}{\Gamma \vdash A}\mathit{cut} \rightsquigarrow \overset{\mathcal{D}}{\Gamma \vdash A}$$

There are two groups of inductive cases, the first is when a cut rule is applied, and then a rule is applied to one of the premises but not on the cut formula. In this case we use rank reduction and try to permute the cut rule above the other rule then apply the inductive hypothesis on smaller l or r.

$$\dfrac{\overset{\mathcal{D}}{\Gamma \vdash C} \quad \dfrac{\overset{\mathcal{E}}{\Gamma, C \vdash A} \quad \overset{\mathcal{F}}{\Gamma, C \vdash B}}{\Gamma, C \vdash A \wedge B}\wedge_R}{\Gamma \vdash A \wedge B}\mathit{cut} \rightsquigarrow \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \vdash C} \quad \overset{\mathcal{E}}{\Gamma, C \vdash A}}{\Gamma \vdash A}\mathit{cut} \quad \dfrac{\overset{\mathcal{D}}{\Gamma \vdash C} \quad \overset{\mathcal{F}}{\Gamma, C \vdash B}}{\Gamma \vdash B}\mathit{cut}}{\Gamma \vdash A \wedge B}\wedge_R$$

We can keep using rank reduction to move cut rules up in the tree until we reach a base case where we can eliminate the cut, or we reach a point where both rules above the cut rule apply on the cut formula. The second inductive case is when a cut rule is applied, and both rules applied on the premises use the cut formula. We use grade reduction to deal with the second, replacing the cut rule that adds C with one or more cut rules that add subformulas of C, then applying the inductive hypothesis on the subformulas of C.

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \vdash A} \quad \overset{\mathcal{E}}{\Gamma \vdash A}}{\Gamma \vdash A \wedge B}\wedge_R \quad \dfrac{\overset{\mathcal{F}}{\Gamma A, B \vdash C}}{\Gamma, A \wedge B \vdash C}\wedge_L}{\Gamma \vdash C}\mathit{cut} \rightsquigarrow \dfrac{\overset{\mathcal{D}}{\Gamma \vdash A} \quad \dfrac{\overset{\mathcal{E}+wk}{\Gamma, A \vdash B} \quad \overset{\mathcal{F}}{\Gamma, A, B \vdash C}}{\Gamma, A \vdash C}\mathit{cut}}{\Gamma \vdash C}\mathit{cut}$$

Repeated applications of the base case, rank reduction, and grade reduction will eventually result in a proof tree that does not use the cut the rule, but the process could cause an exponential increase in the size of the proof tree.

The web application can be accessed at:
https://logic.qatar.cmu.edu/sequoia/
The project's source code is available on GitHub:
https://github.com/meta-logic/sequoia

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